Complex resonant ice shelf vibrations

Luke Bennetts, Uni Adelaide

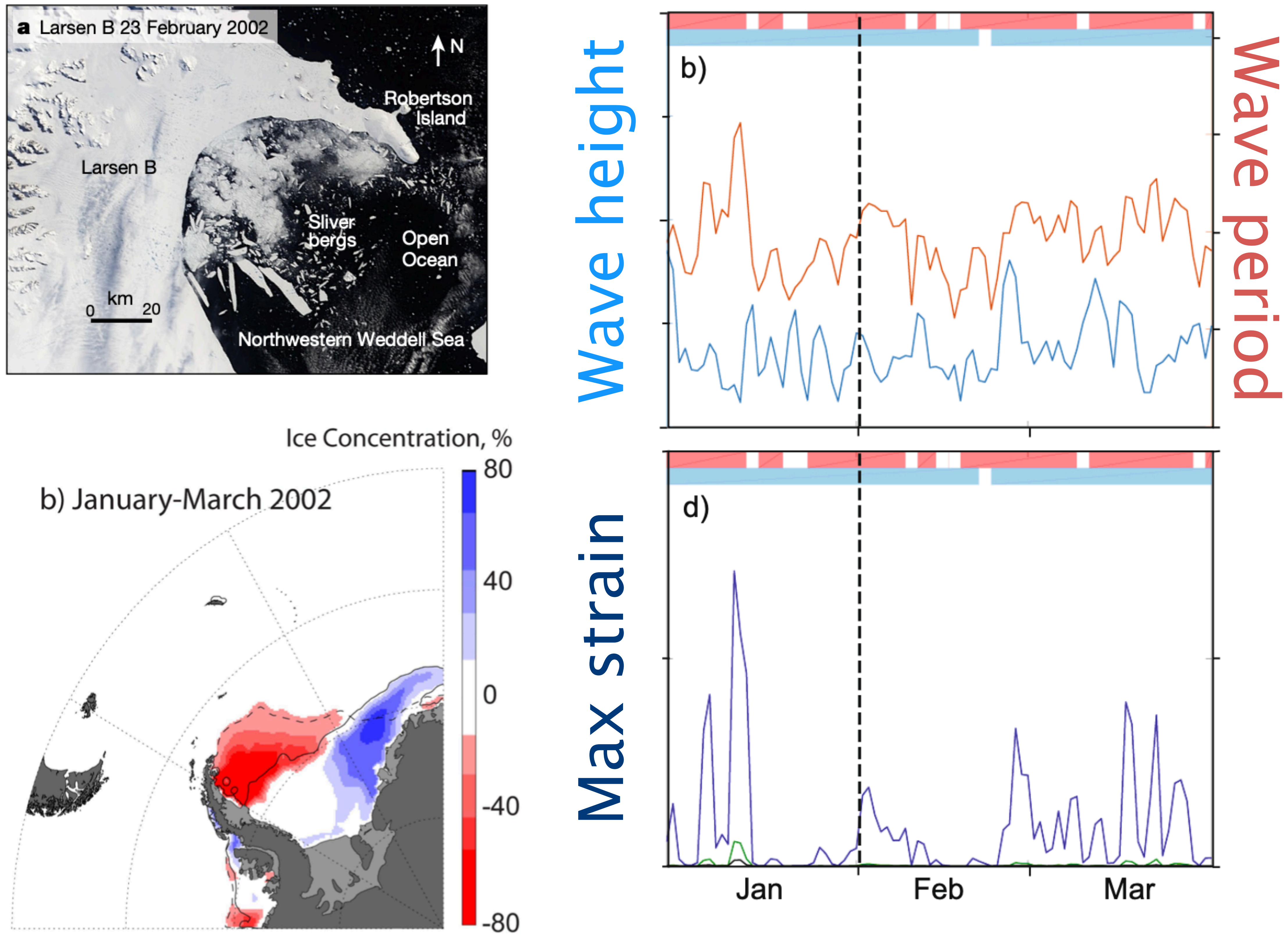
Mike Meylan, Uni Newcastle

Australian Government
Australian Research Council

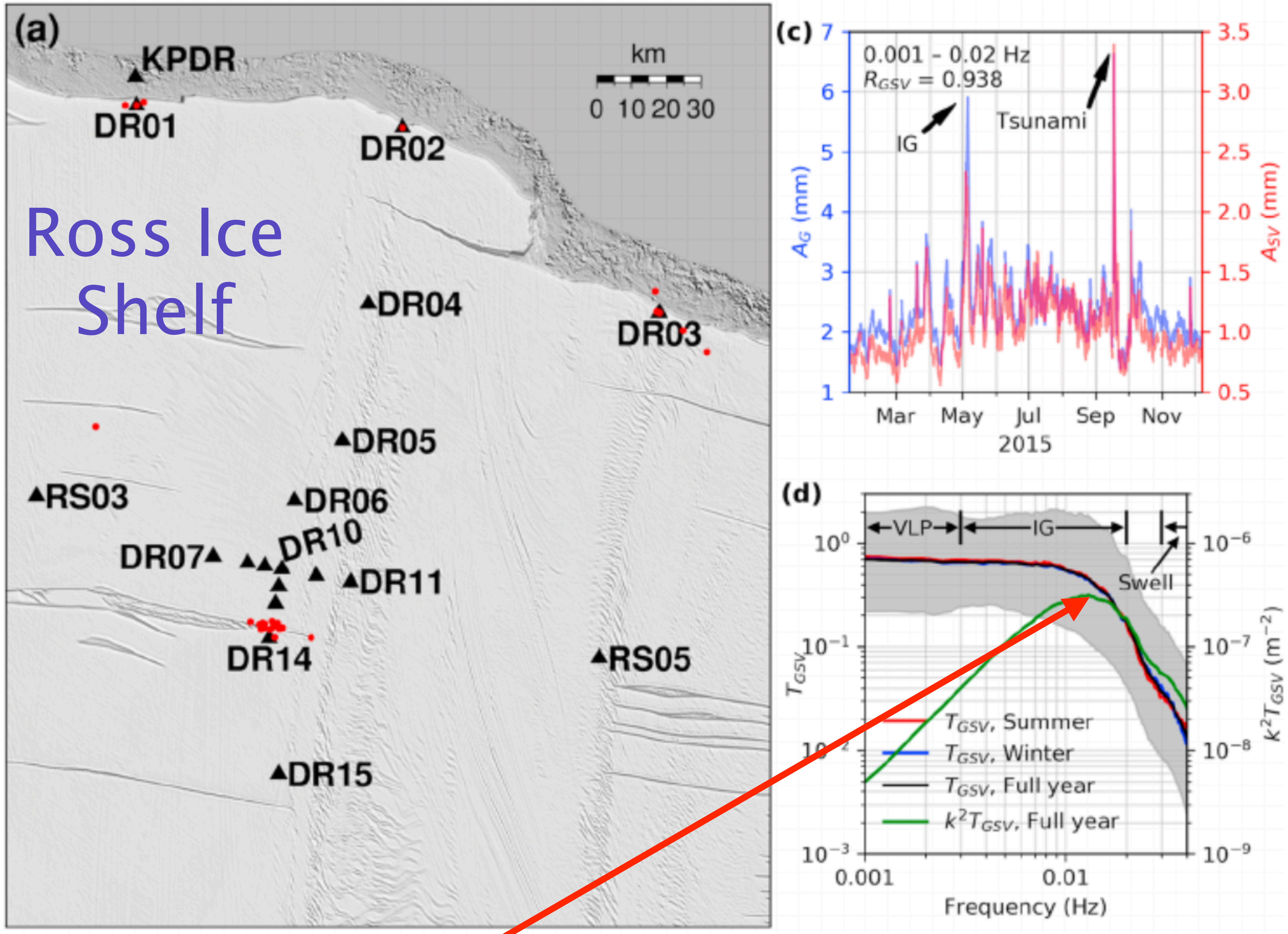
THE UNIVERSITY of ADELAIDE
SUB CRUCE LUMEN

Australian Government
Department of Sustainability, Environment, Water, Population and Communities
Australian Antarctic Division

Motivation: wave-induced shelf vibration measurements



Massom et al, Nature, 2018



Max strain at period 50–100s or $\omega/\pi = 0.02$–0.04 Hz

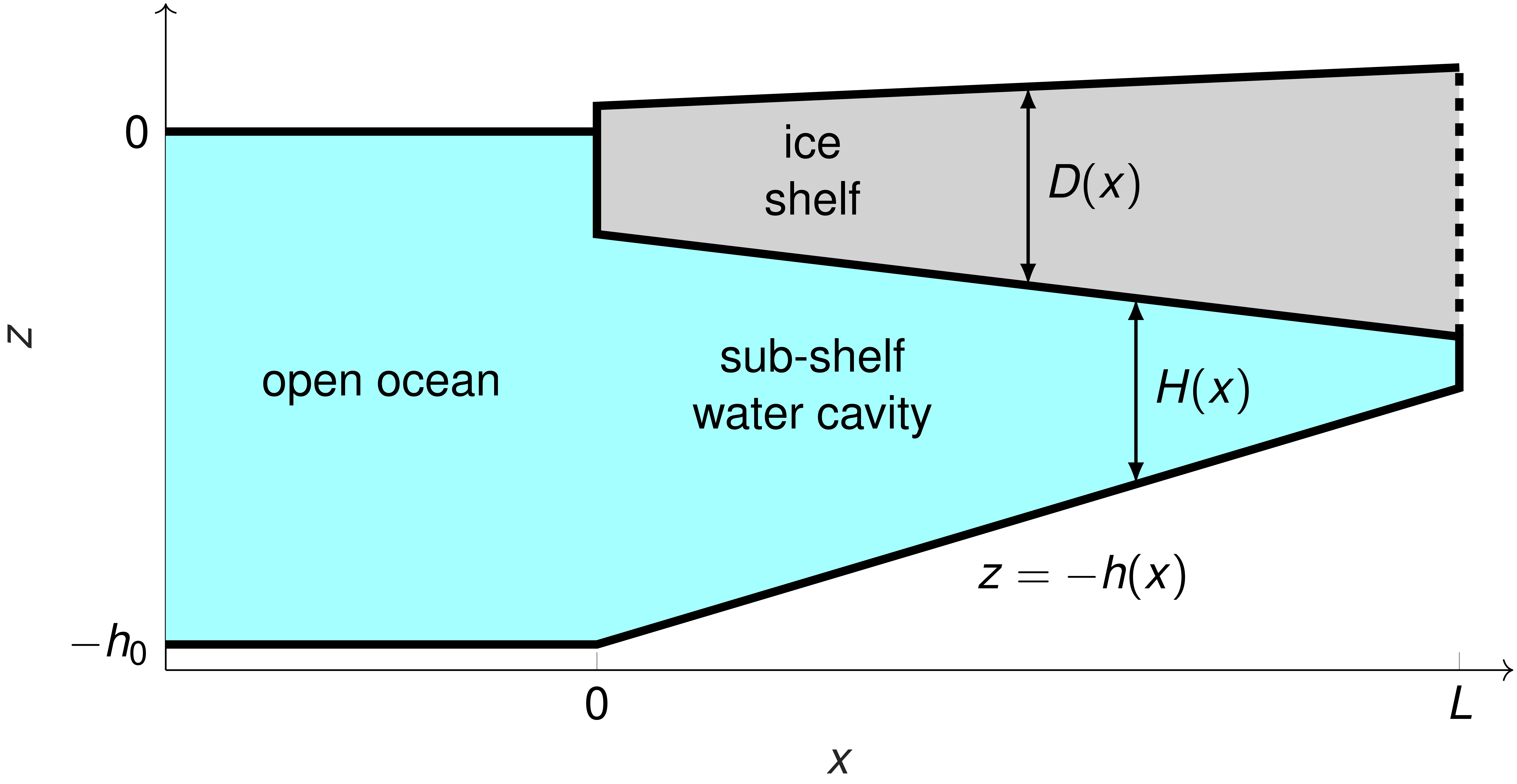
Chen et al, Geophy Res Lett, 2019

Standard model

Potential flow: shallow water or finite depth

Kalyanaramen et al, Wave Motion, 2019

Model with thickening shelf and shoaling seabed



Governing equations

Single-mode approximation

- Water velocity field is real part of

$$\mathrm{Re}\{\phi(x,z)\,\mathrm{e}^{-\mathrm{i}\omega t}\} \quad \text{where} \quad \begin{cases} \omega \in \mathbb{R}^+ & \text{angular velocity (prescribed)} \\ \phi \in \mathbb{C} & \text{velocity potential (unknown)} \end{cases}$$

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- In open ocean

$$\phi(x,z) \approx \varphi_0(x)\,\frac{\cosh\,k(z+h_0)}{\cosh(k\,h_0)} \quad \text{where} \quad k\,\tanh(k\,h_0) = \sigma \equiv \frac{\omega^2}{g}$$

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- In cavity

$$\phi(x,z) \approx \varphi(x)\,\frac{\cosh\,\kappa(z+h)}{\cosh(\kappa\,h)} \quad \text{where} \quad (1-\sigma\,d+\Gamma\,\kappa^4)\,\kappa\,\tanh(\kappa\,H) = \sigma$$

Governing equations

Depth averaged equations

- In open ocean, set

$$\varphi_0(x) = A_{\text{inc}} \left(e^{i k x} + R e^{-i k x} \right)$$

where A_{inc} is incident amplitude (prescribed), and $R \in \mathbb{C}$ is the reflection coefficient (unknown).

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- In shelf/cavity, solve ODE system
$$(a\,\varphi')' + b\,\varphi + \sigma\,\zeta = 0 \quad \text{and} \quad (1 - \sigma\,d)\,\zeta + \mathcal{L}\{\zeta\} - \varphi = 0$$
with known coefficients $a(x)$ and $b(x)$, and where $\mathrm{Re}\{\zeta(x)\,\mathrm{e}^{-\mathrm{i}\,\omega\,t}\}$ is the shelf vibration (unknown).

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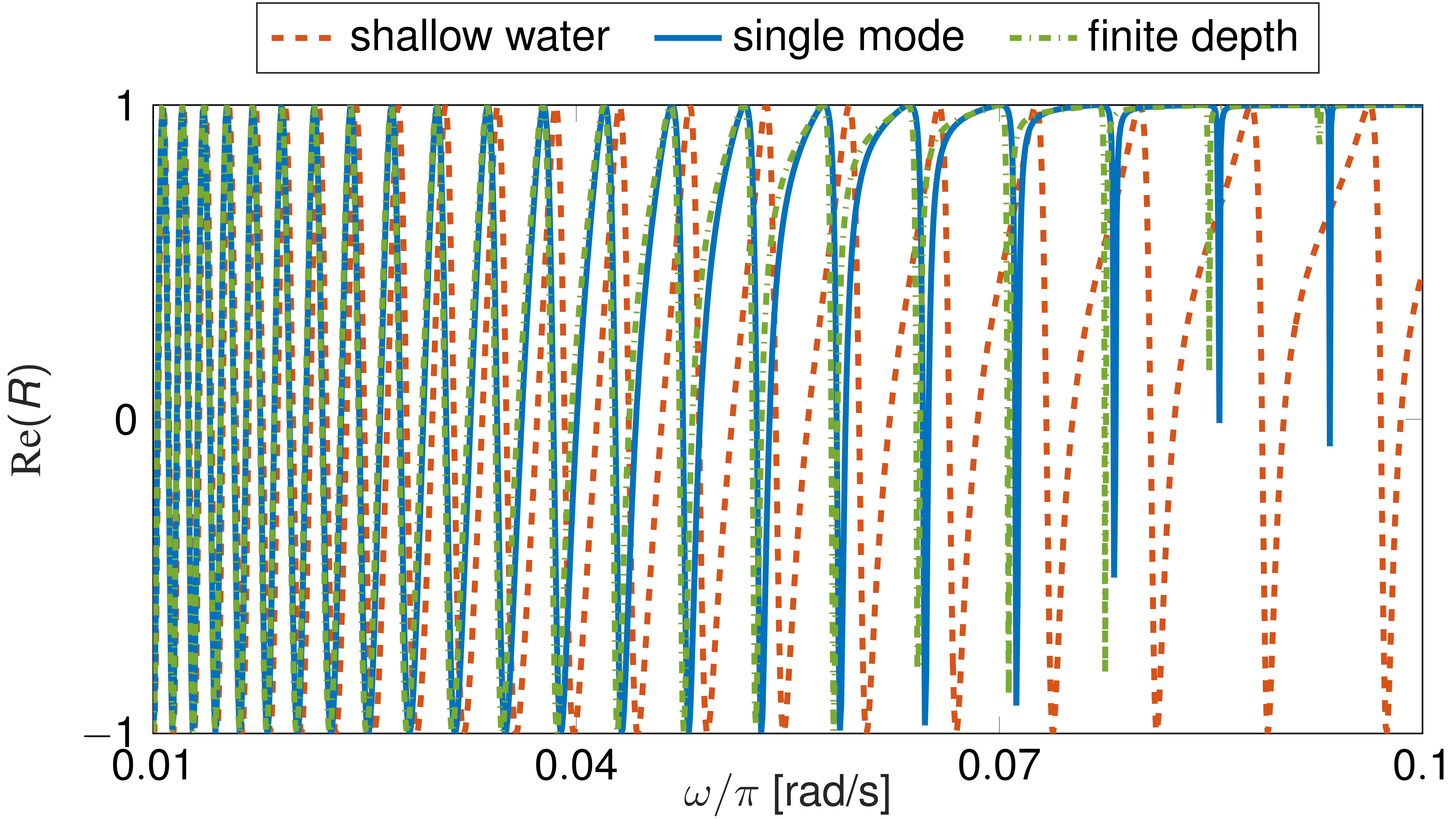
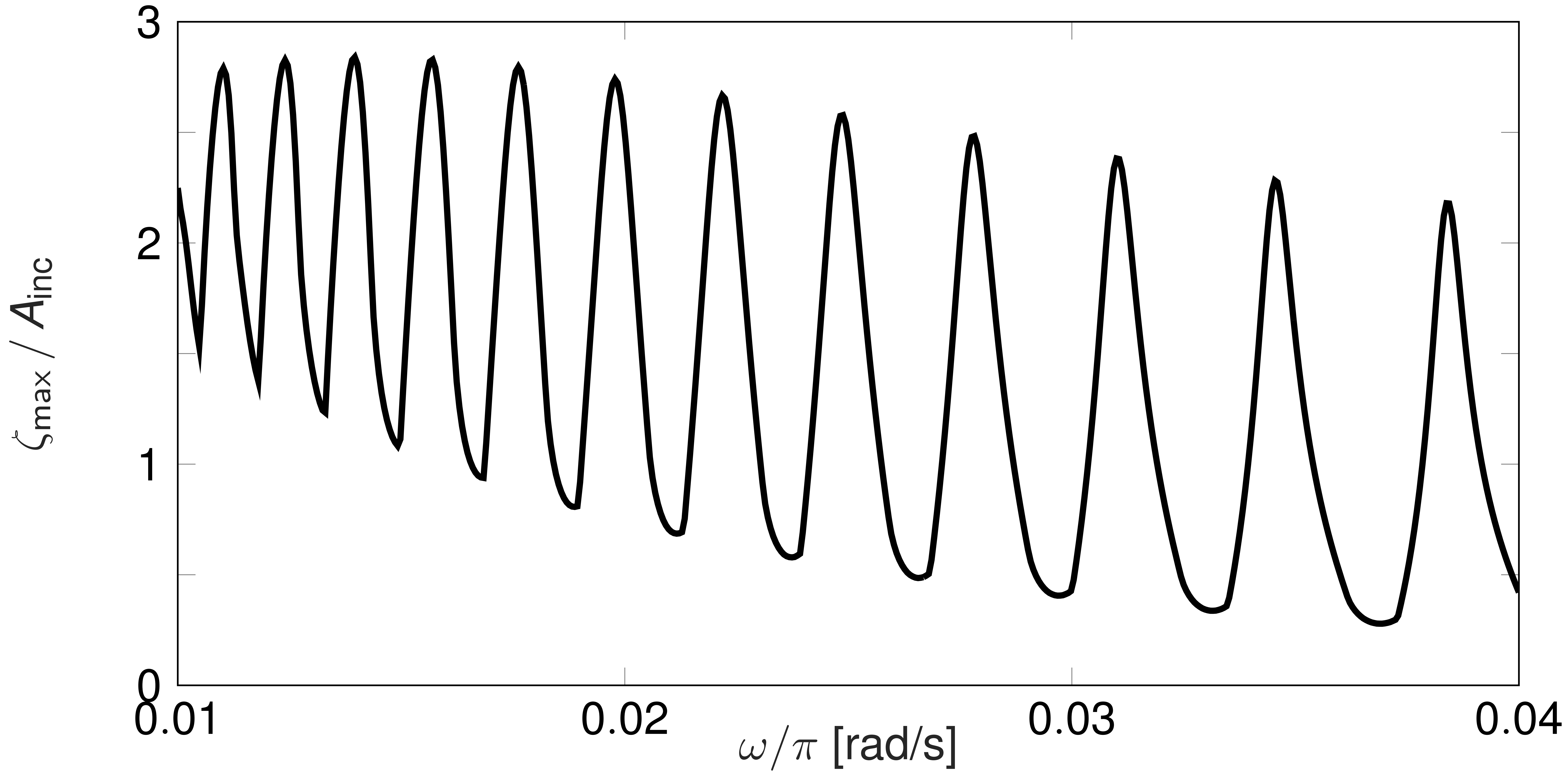
- \+ “jump conditions" at $x = 0$, i.e. depth averaged continuities.
- \+ shelf end conditions, i.e. free at $x = 0$ and clamped at $x = L$.

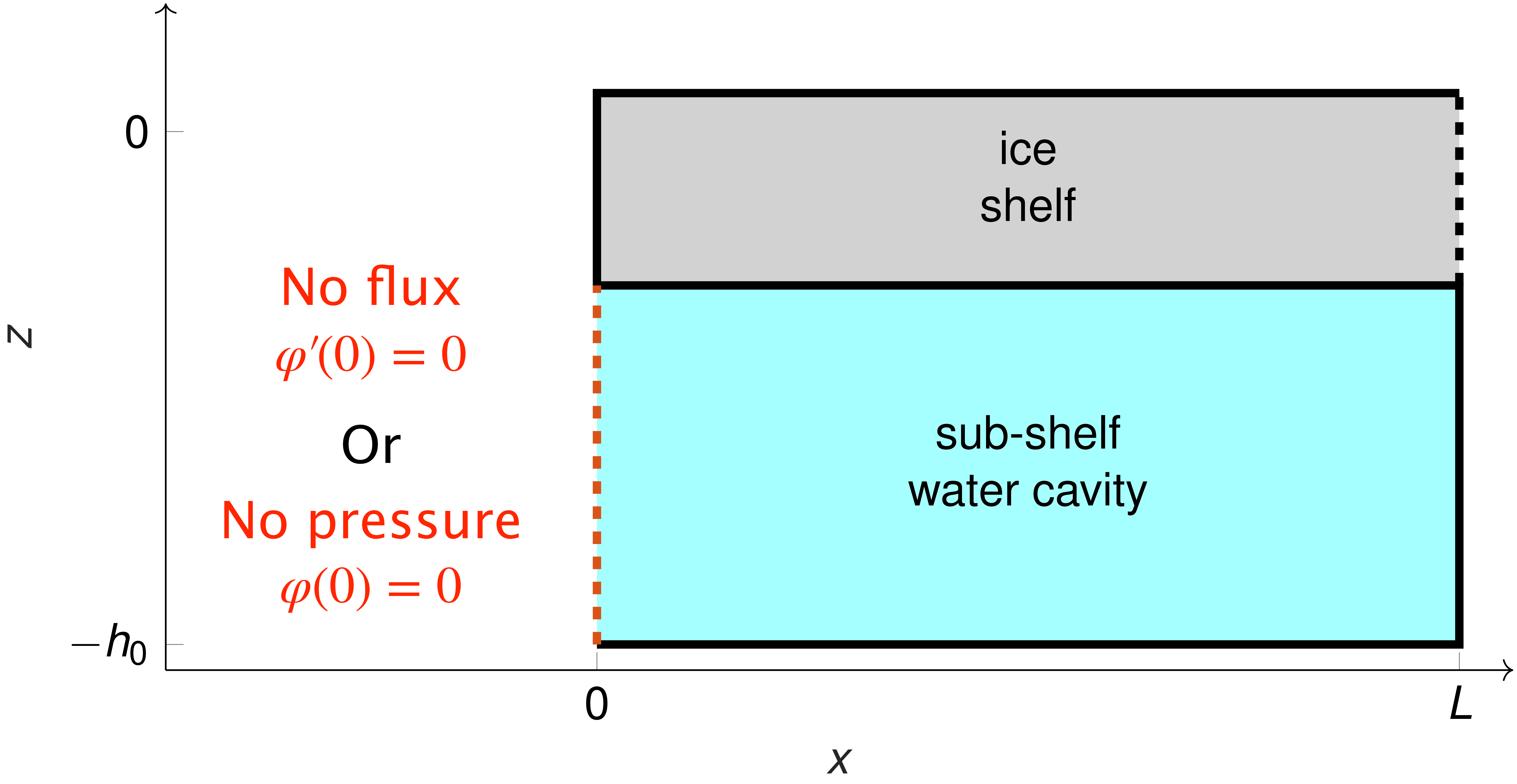
Accuracy of single-mode approximation (uniform geometry)



Maximum shelf displacement: Uniform geometry



Uncoupled problems



Sergienko, J Glaciol, 2013; Meylan et al, J Glaciol, 2017; Papathanasiou et al, Ocean Model, 2019

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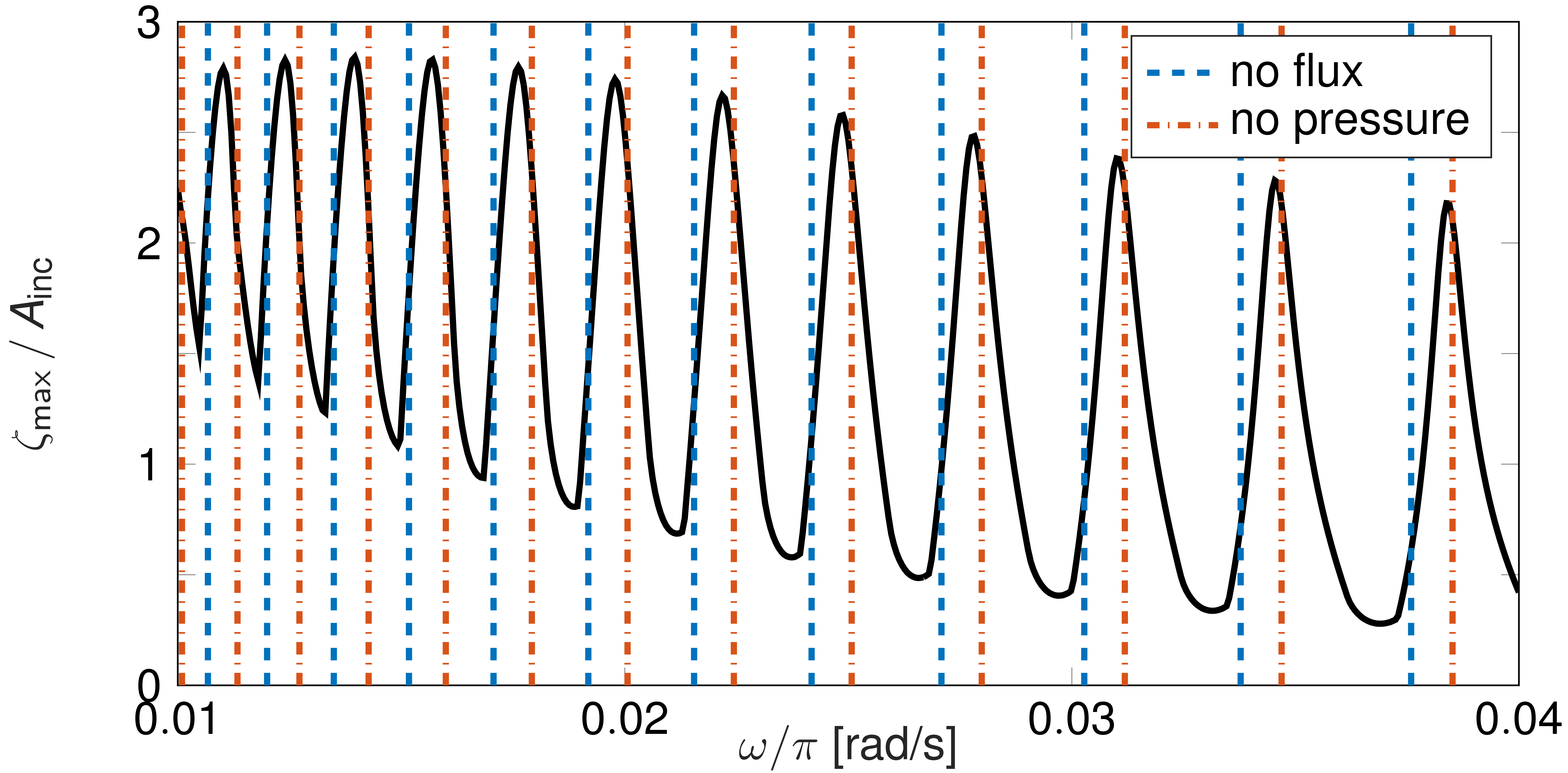

Jump conditions

- Can be expressed as

$$c_1 \left(\frac{\varphi'(0)}{\kappa} \right) + \mathrm{i}\, c_2\, \varphi(0) = 2\,\mathrm{i}\, k\, A_{\mathrm{inc}}$$

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- Resonance if non-zero solution for $A_{\text{inc}} = 0$, i.e.

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- Closer to no-flux condition if $C \ll 1$ and no-pressure condition if $C \gg 1$.

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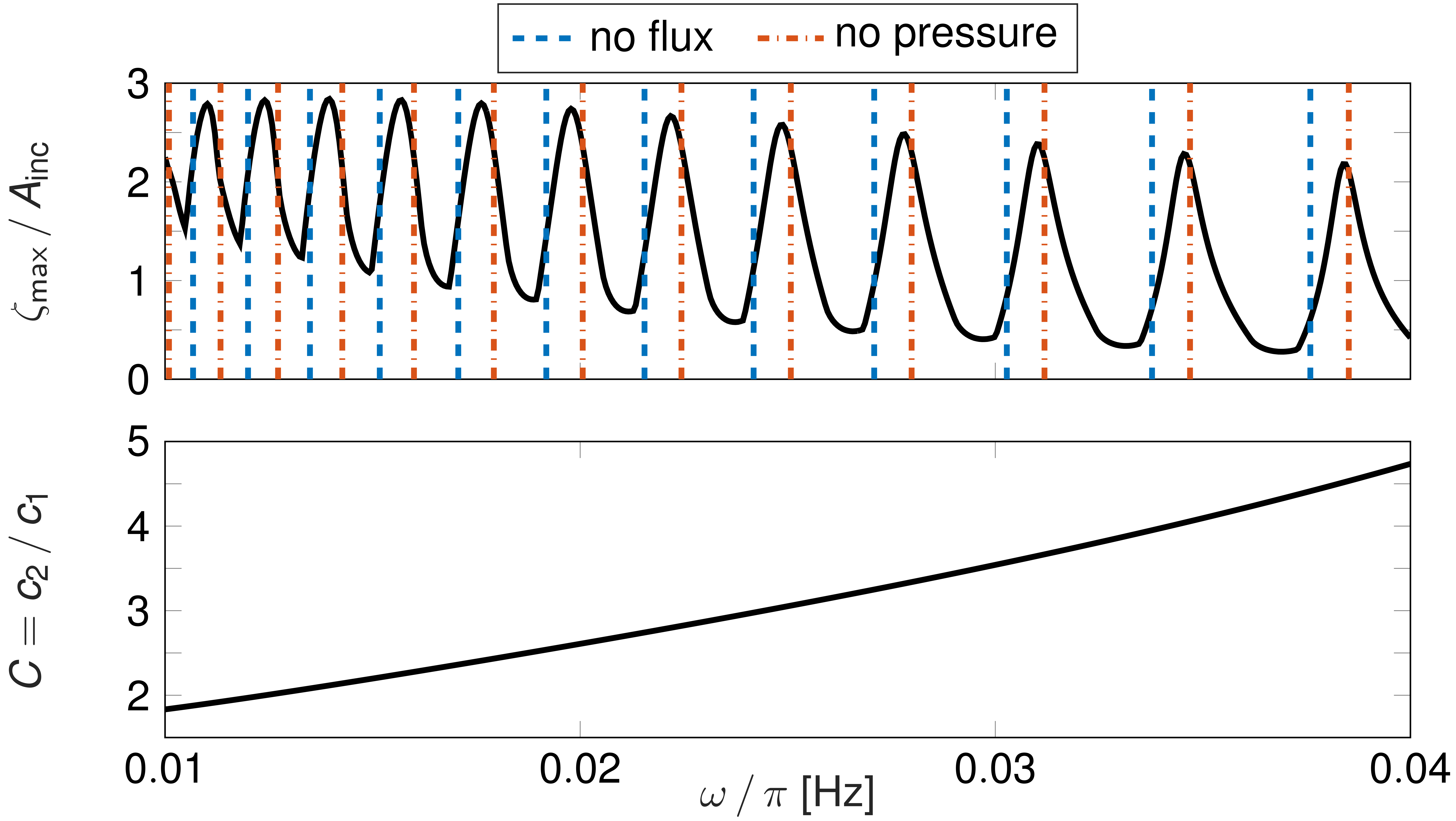

Complex resonances

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- Uncoupled eigenfrequencies $\in \mathbb{R} \Rightarrow$ easy/cheap to calculate.
- Complex frequencies $\in \mathbb{C} \Rightarrow$ difficult/expensive to calculate.

Homotopy method

- Find ω_m from $\det(\mathcal{M}) = 0$, where $\mathcal{M}$ is 6×6 matrix

$$\mathcal{M}(\omega) = \begin{pmatrix} \mathcal{I} & -\mathcal{R}_{+}(\omega)\,\mathcal{E}(\omega) \\ -\mathcal{R}_{\mathrm{ld}}(\omega)\,\mathcal{E}(\omega) & \mathcal{I} \end{pmatrix}$$

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- Uncoupled eigenfrequencies satisfy similar relation, but with $\mathcal{R}_+ \mapsto \mathcal{R}_{\mathrm{f}}$ or $\mathcal{R}_{\mathrm{p}}$:

$$\mathcal{R}_{\mathrm{f}} = \mathcal{R}_+ + \mathcal{T}_- \left(1 - \mathcal{R}_-\right)^{-1} \mathcal{T}_+ \quad \text{and} \quad \mathcal{R}_{\mathrm{p}} = \mathcal{R}_+ - \mathcal{T}_- \left(1 + \mathcal{R}_-\right)^{-1} \mathcal{T}_+$$

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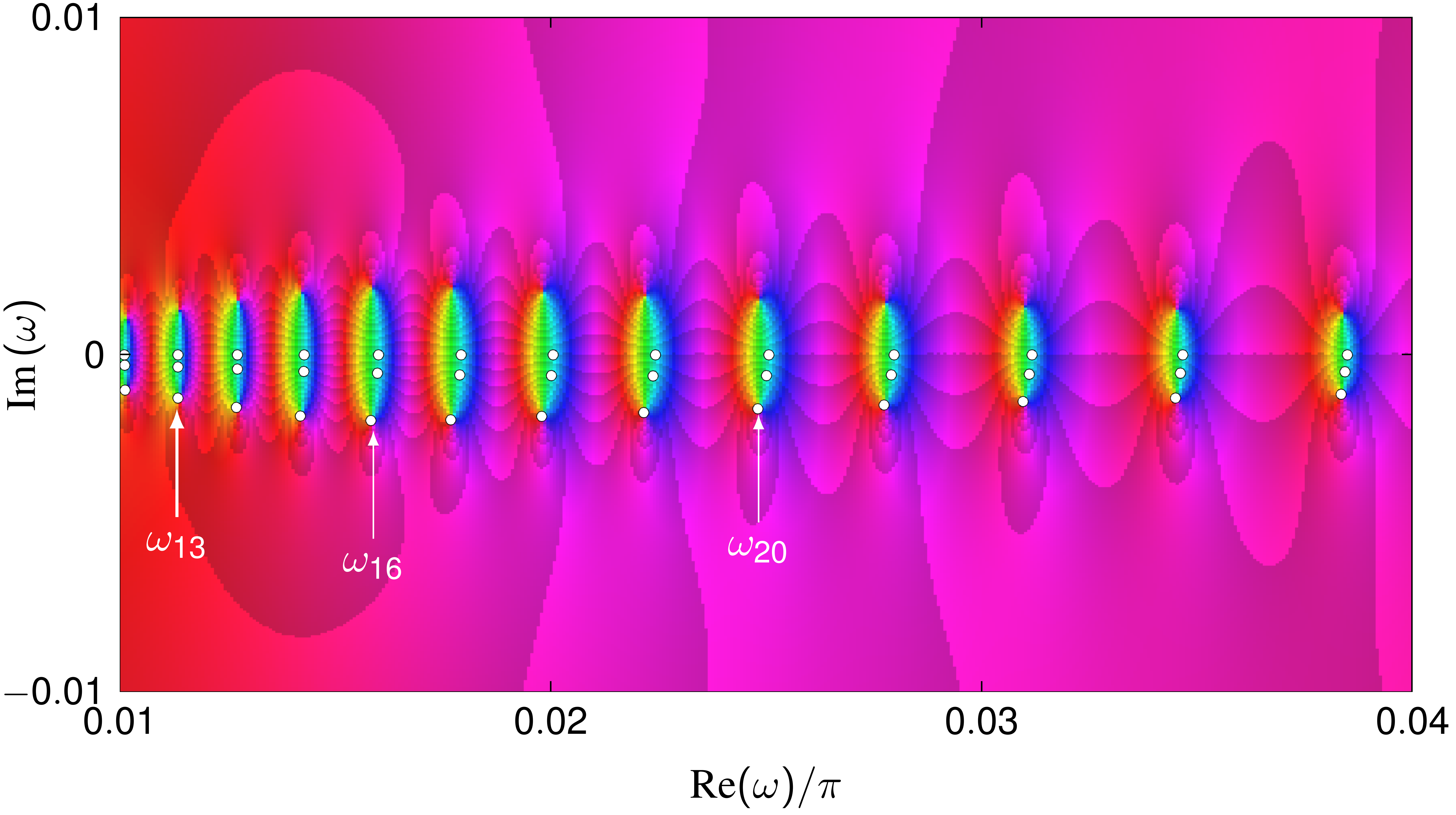
- Construct homotopy in which $\mathcal{R}_+ \mapsto \mathcal{R}_\hbar$:

$$\mathcal{R}_\hbar = \mathcal{R}_+ + (1-\hbar)\,\mathcal{T}_- \left(1 - \mathcal{R}_-\right)^{-1} \mathcal{T}_+ \quad \text{or} \quad \mathcal{R}_+ - (1-\hbar)\,\mathcal{T}_- \left(1 + \mathcal{R}_-\right)^{-1} \mathcal{T}_+$$

- Start with eigenfrequencies and vectors for uncoupled problem (no flux or no pressure) and solve iteratively, e.g. for $\hbar = 0, 0.1, 0.2, \ldots, 1$.

Reflection coefficient in complex frequency space



Blaschke product

- Note that

$$|R|^2 = 1 \quad \text{for} \quad \omega \in \mathbb{R} \quad \text{i.e. energy conservation}$$

and $\quad R(\overline{\omega}) = |R(\omega)|^{-1}\, \mathrm{e}^{\mathrm{i}\,\arg\{R(\omega)\}}.$

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$$\text{and} \quad R(\overline{\varpi}) = |R(\omega)|^{-1}\, \mathrm{e}^{\mathrm{i}\, \arg\{R(\omega)\}}.$$

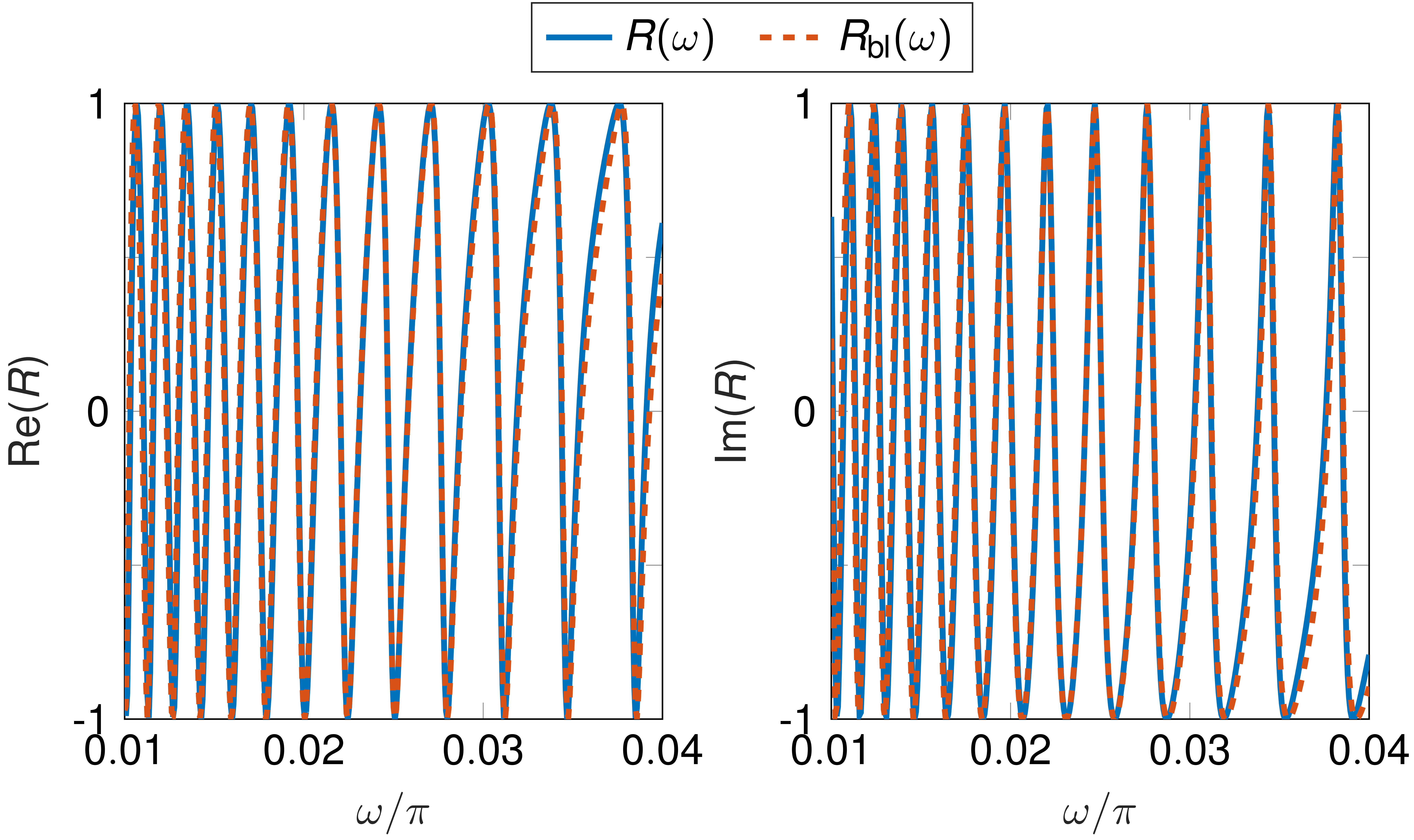
- Define

$$R_{\mathrm{bl}}(\omega) = \prod_{n=1}^{\infty} r(\omega : \omega_n)\, r(\omega : -\overline{\omega_n})$$

where

$$r(\omega : \varpi) = \frac{\omega - \overline{\varpi}}{\omega - \varpi}.$$

Blaschke product



Incident wave packets and singularity expansion method

Gaussian incident packet

- Defined by the Fourier transform (in k)

$$\mathcal{F}\{u_{\text{inc}}\} = \frac{1}{\pi}\sqrt{2\,\beta}\,\mathrm{e}^{-\beta\,(k-k_0)^2}$$

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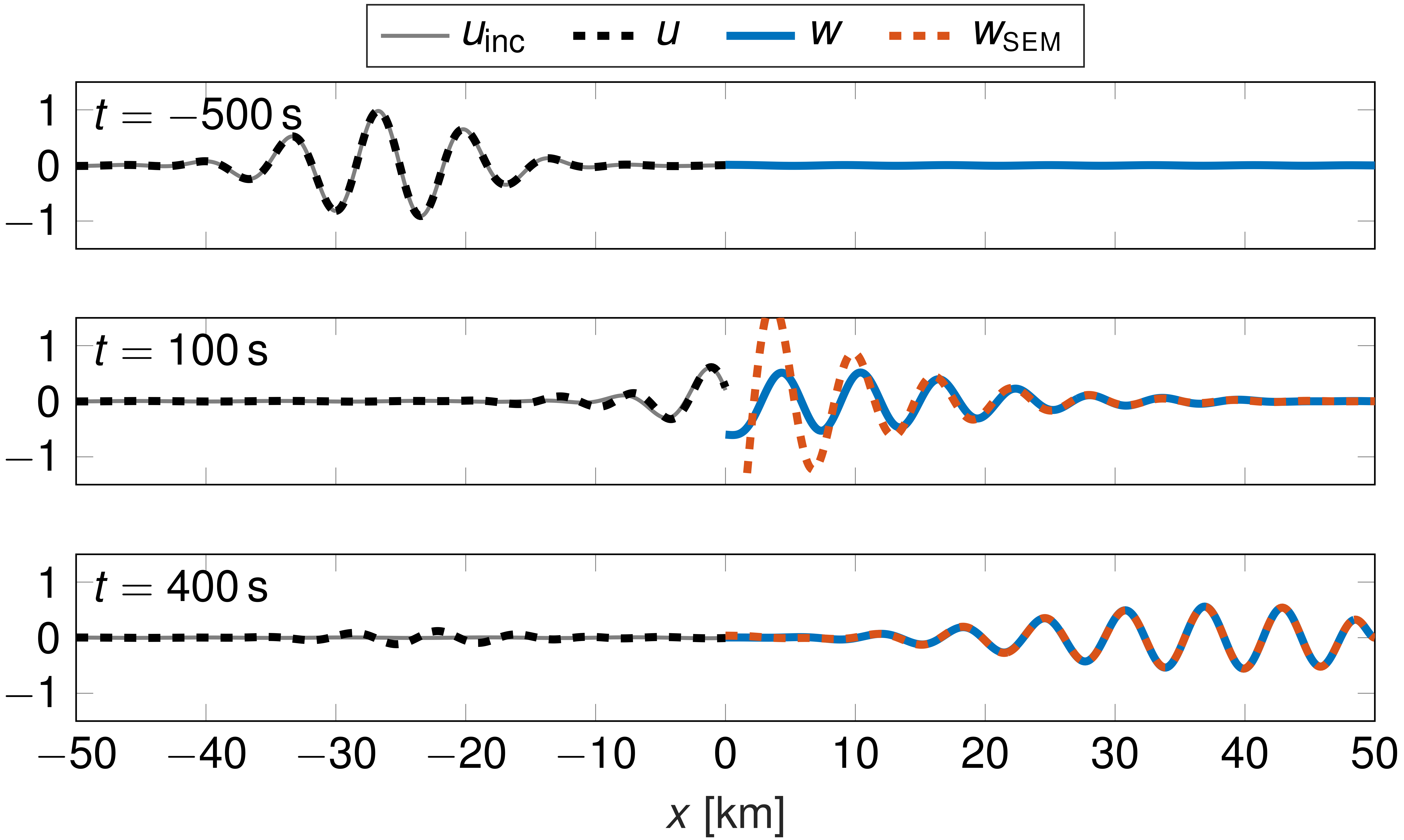
Singularity expansion method

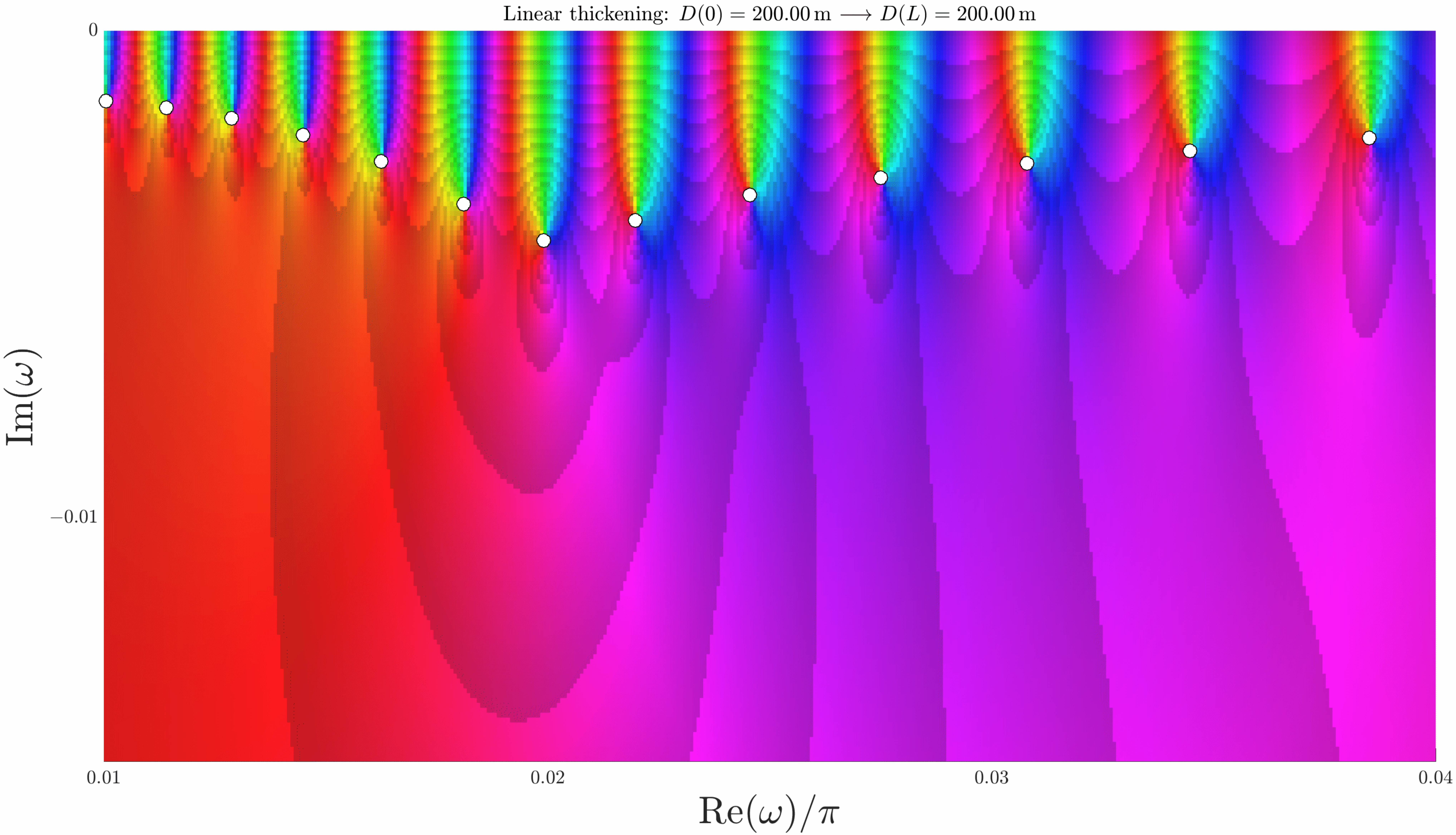
- For long times, the shelf displacement

$$w(x,t) \sim w_{\text{SEM}}(x,t) = \sum_{n=1}^{\infty} w_n(x,t) \quad \text{where} \quad w_n = \mathrm{Re}\{A_n\,\mathrm{e}^{-\mathrm{i}\,\omega_n\,t}\}.$$

Example time domain simulation



Thickening shelf: $R(\omega)$



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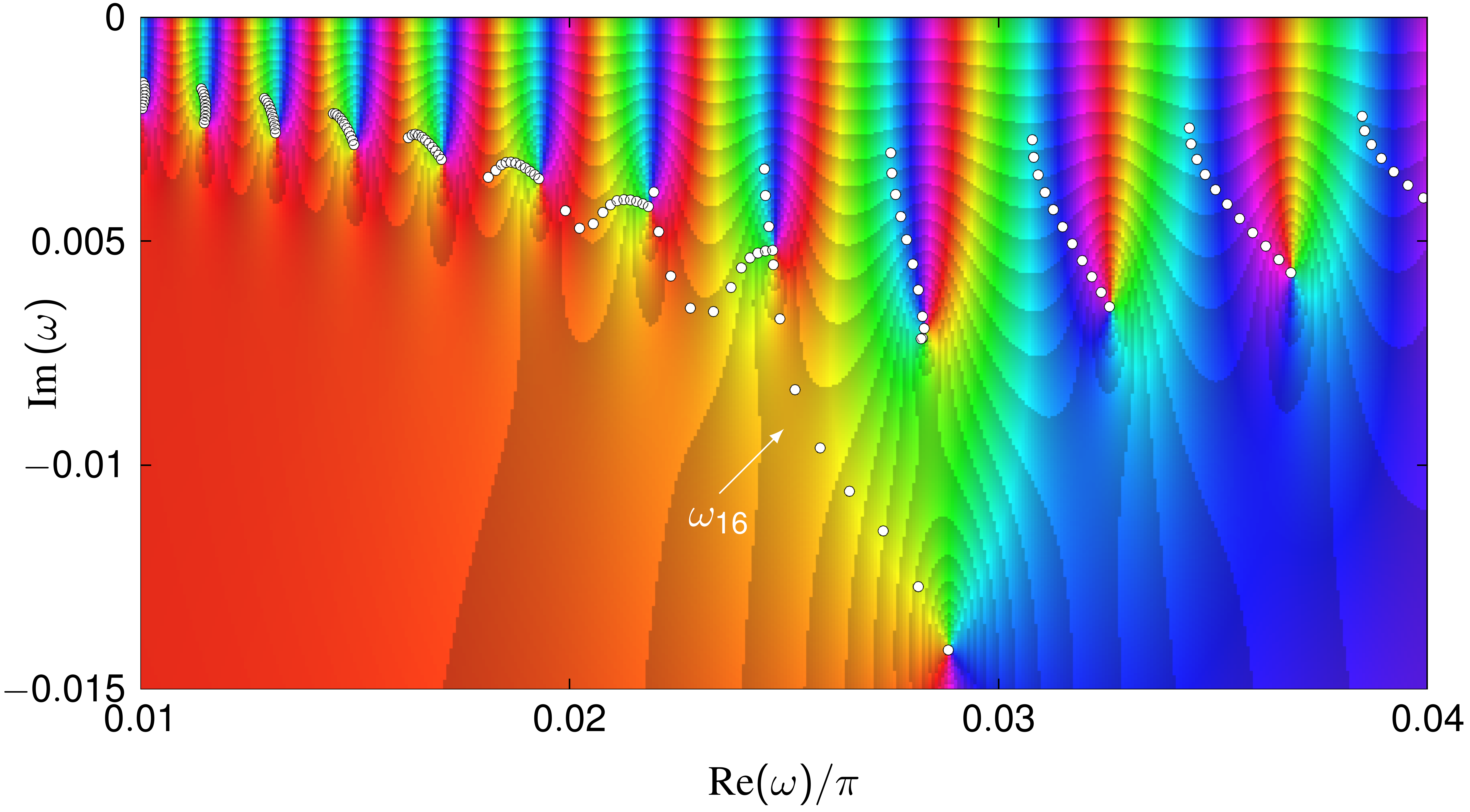
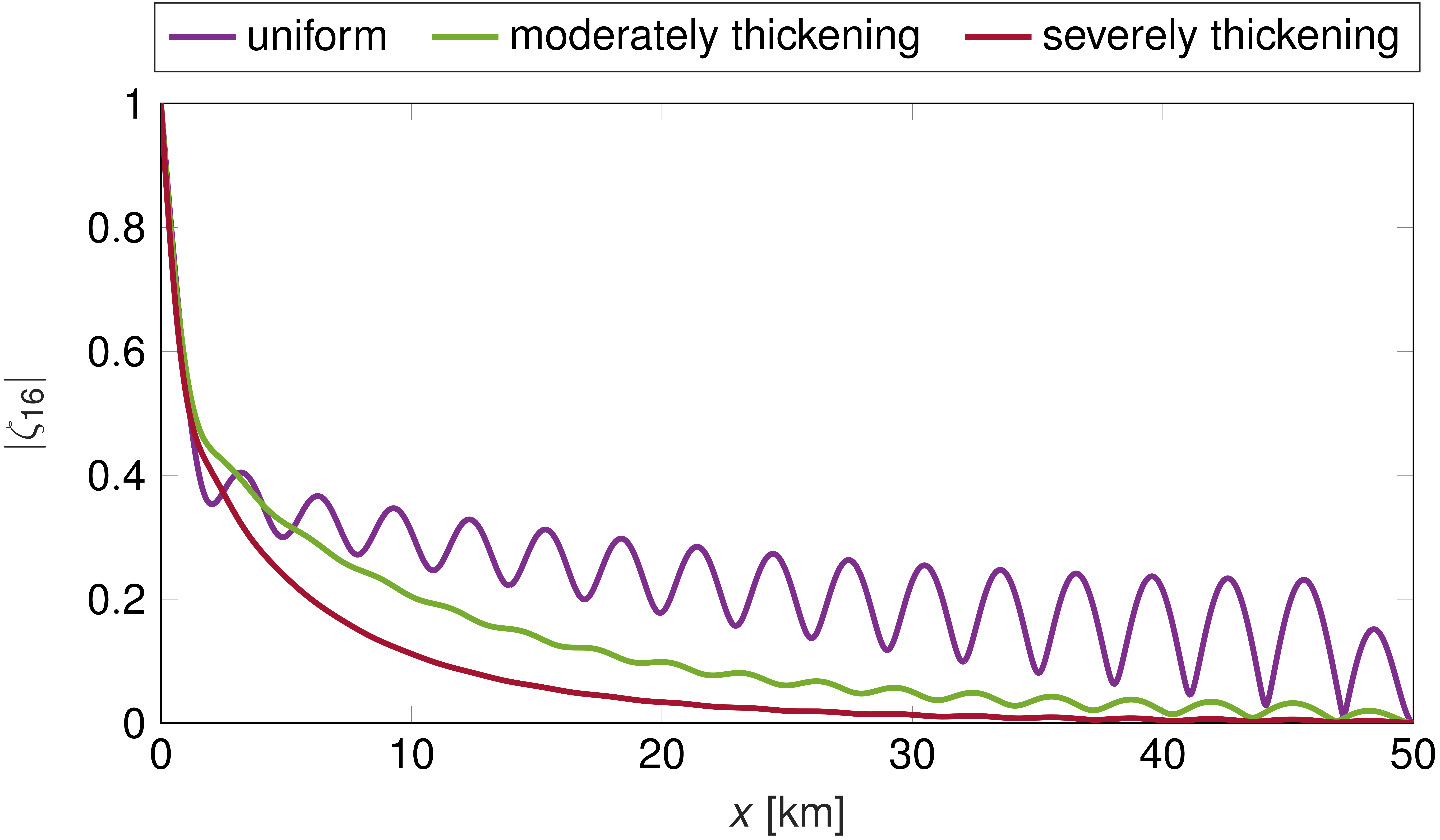


Thickening shelf: complex resonant modes



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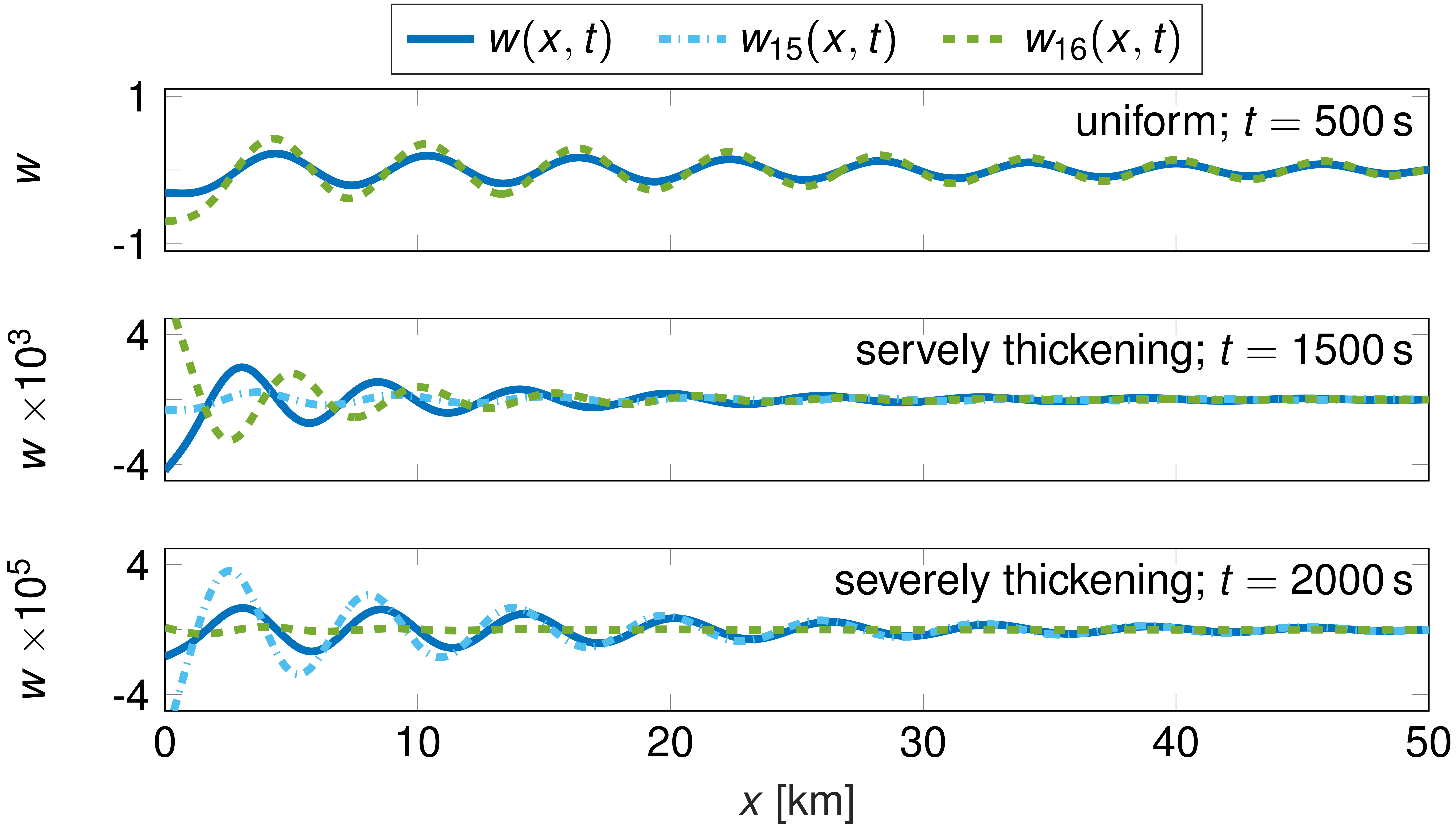


Summary

Methods

- Efficient method for non-uniform geometries.
- Homotopy method to find complex resonances.

Complex reonances

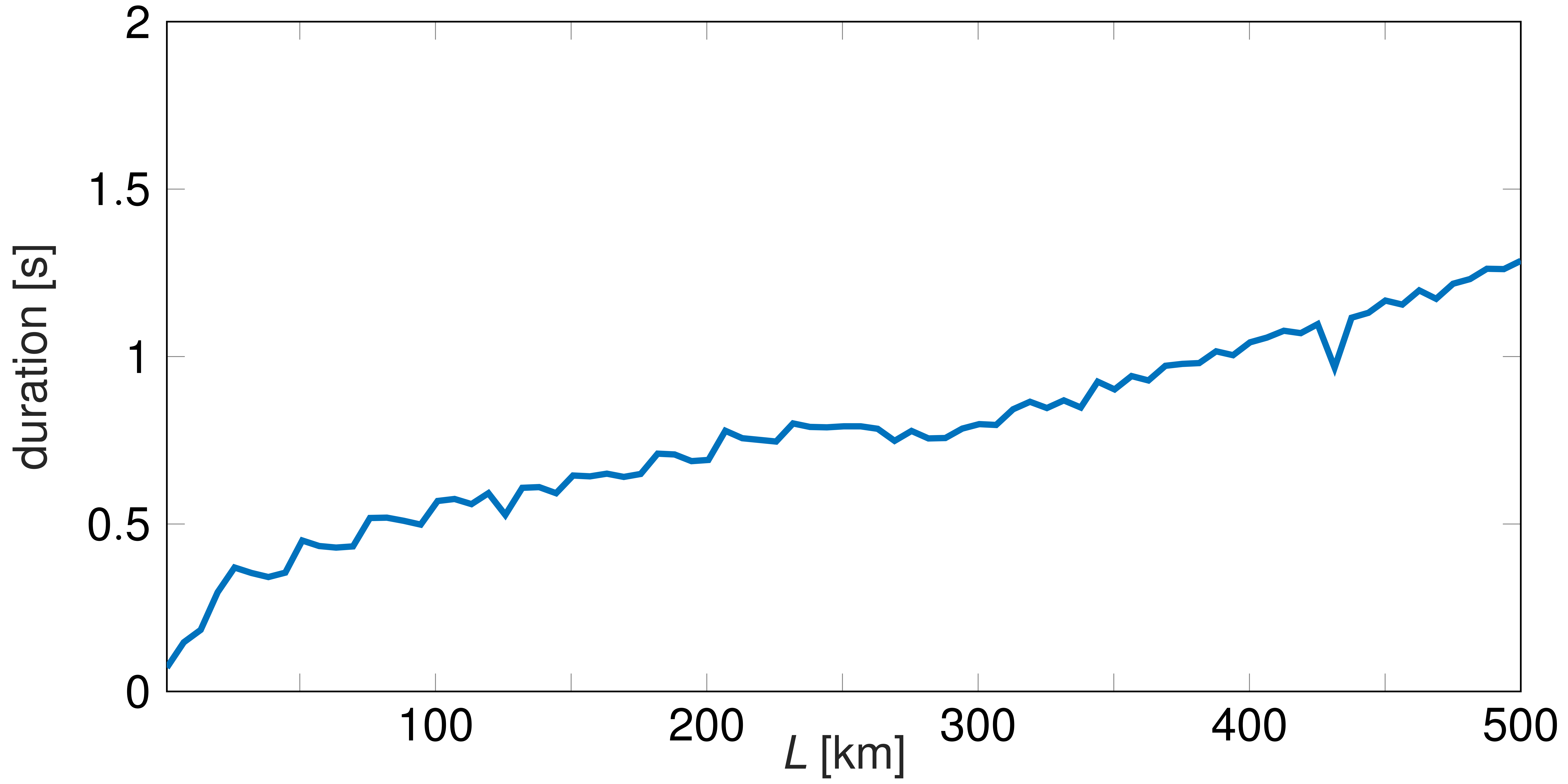
- Approximate frequency-domain solutions via Blaschke product.
- Capture long-time behaviour of transient solutions.

Thickening shelf

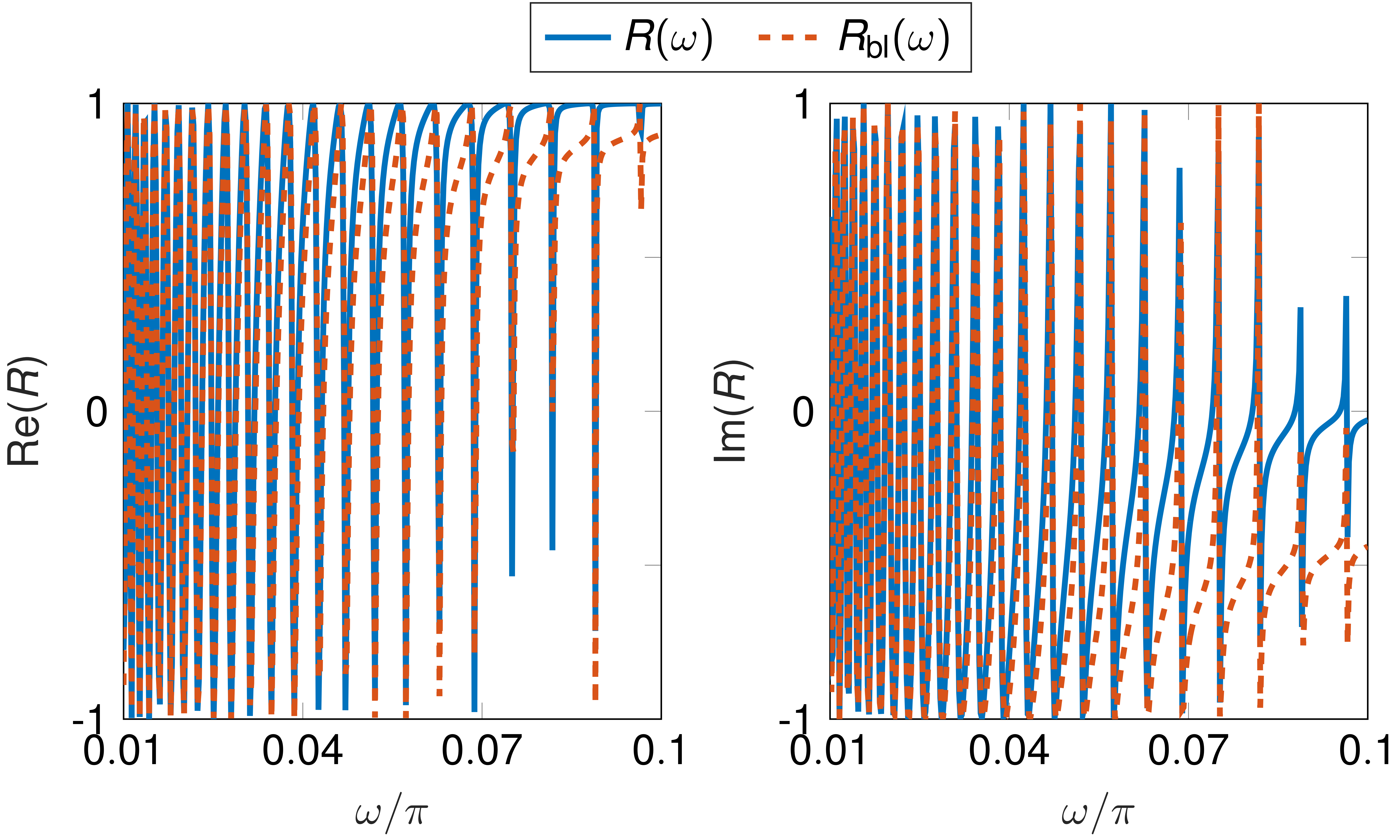
- Can prevent mid-range-frequency resonances from being excited.

Coda

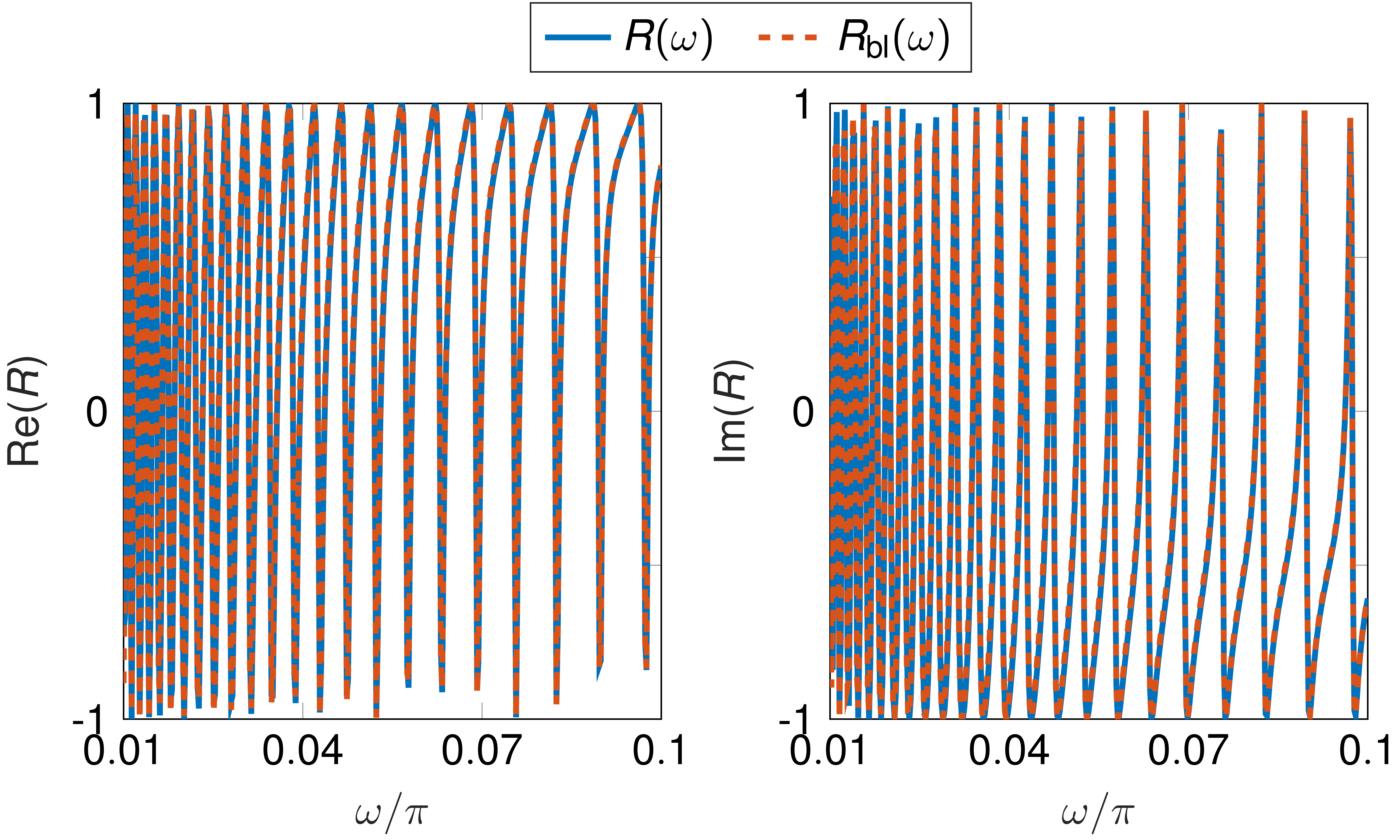
Efficiency of step approximation for varying geometry



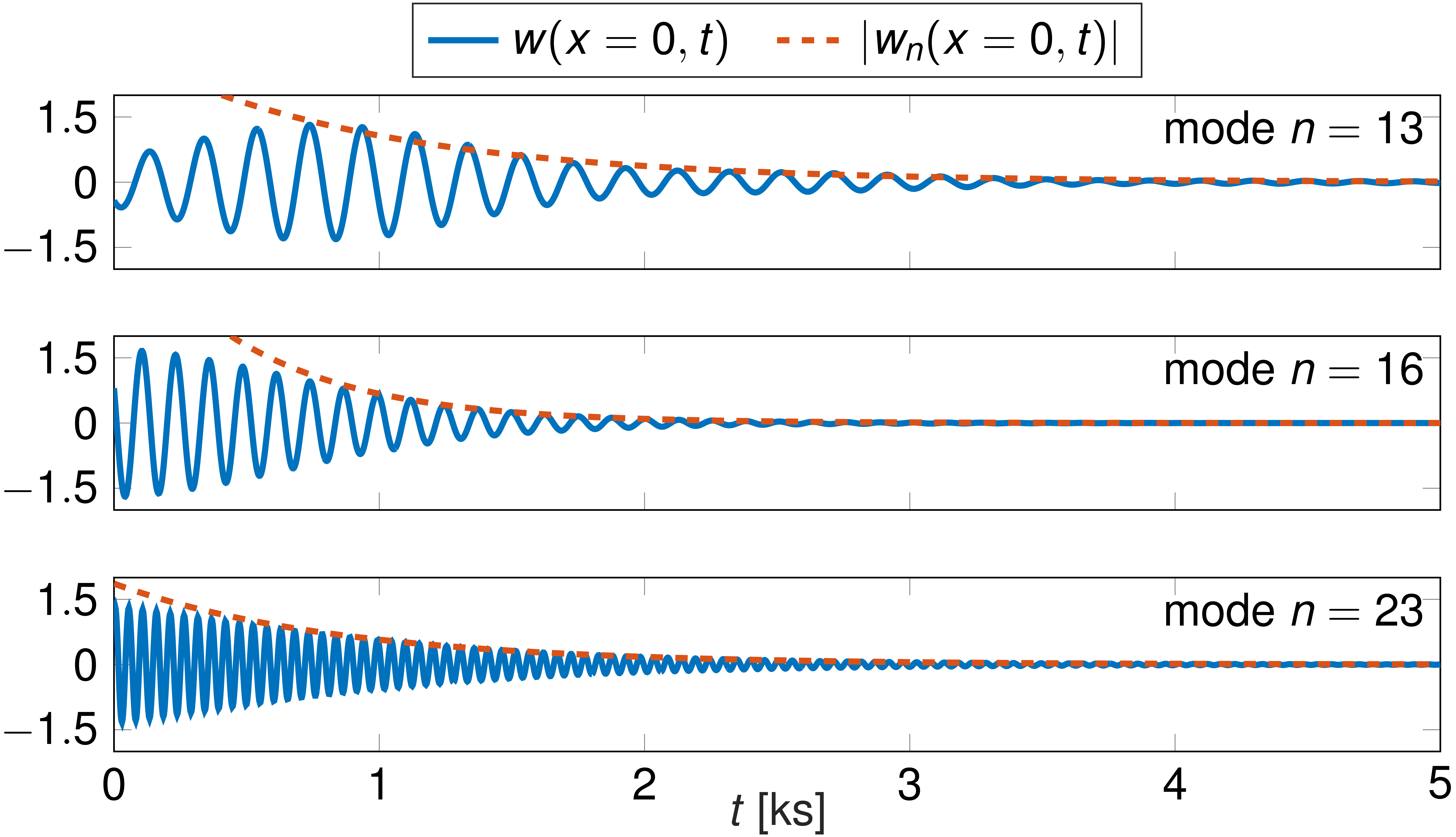
Blaschke product: Extended frequency range



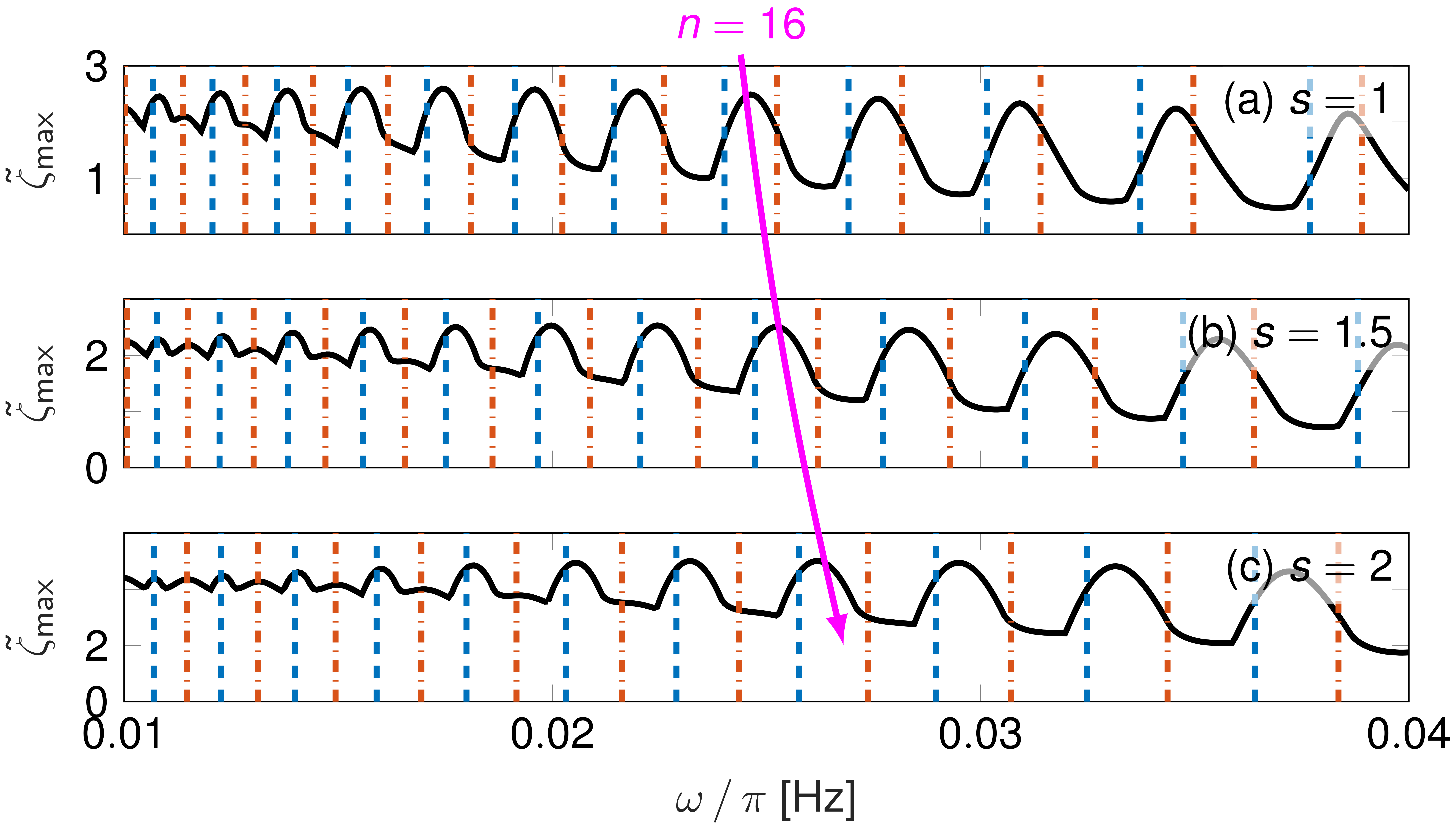
Blaschke product: Shallow-water approximation



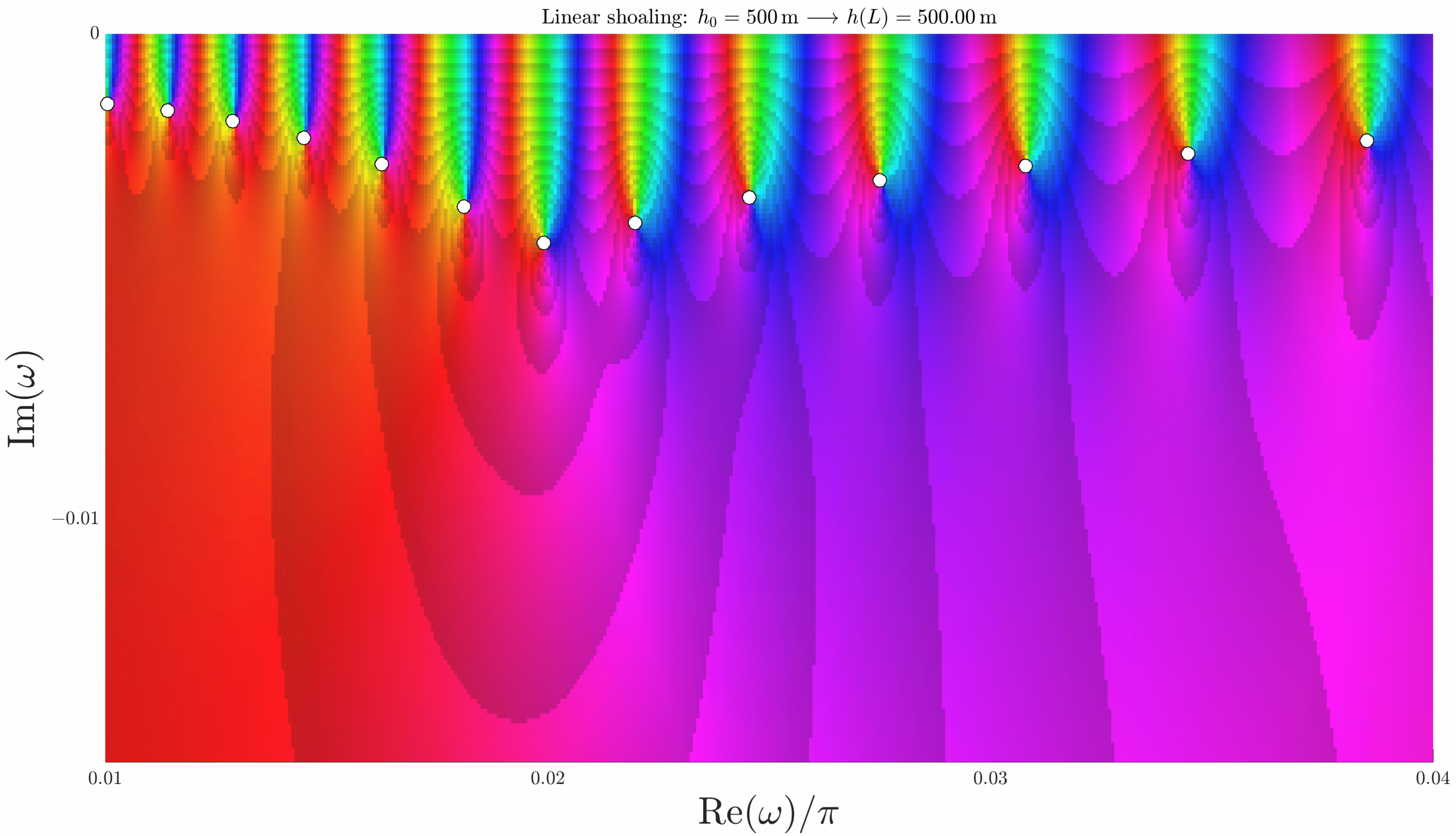
Resonant lifetimes



Maximum shelf displacement: Thickening shelf



Shoaling bed: $R(\omega)$



Thickening shelf: complex resonant modes η_n